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High School Students' Combinatorial Thinking in Solving Combinatoric Problems Based on Mathematical Ability

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Abstract: The purpose of this research is to describe the combinatorial thinking of high school students in solving combinatoric problems based on mathematical ability. Combinatorial thinking is a basic thinking ability that must be continuously developed towards critical thinking abilities and skills, so as to build one's knowledge or arguments and experiences. This research is a descriptive research using a qualitative approach. The research participants consisted of three 16-year-old students who had studied probability material for class X and had high, medium, and low mathematical abilities. The data in this research were obtained through combinatoric problem assignments and task-based interviews. The data obtained will be analyzed by reducing data, presenting data, and drawing conclusions. The results of the research show that: (a) high-ability students' combinatorial thinking starts from formulas/expressions, counting processes, sets of outcomes, expressions, counting processes, sets of outcomes, counting processes, then sets of outcomes in sequence, which fulfills all indicators of the level of combinatorial thinking and using two types of verification strategies, (b) medium-ability students' combinatorial thinking starts from expressions, sets of outcomes, formulas, counting processes, sets of outcomes, counting processes, then sets of outcomes in sequence, which fulfills all indicators of the level of combinatorial thinking and uses one type of verification strategy, (c) low-ability students' combinatorial thinking starts from expressions, sets of outcomes, counting processes, and then sets of outcomes in which some indicators of the level of combinatorial thinking are met and do not use verification strategies.

INTRODUCTION

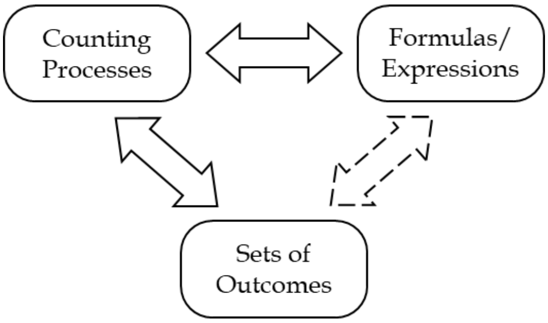
Combinatorial thinking is very important to develop, because combinatorial thinking is a basic thinking ability that must continue to be developed towards critical thinking skills, so that it can build one's knowledge or arguments and experience (Hidayati et al., 2020). Combinatorial thinking includes how one tries to calculate several possibilities in a combination of several objects from the available objects (Uripno & Rosyidi, 2019). Hidayati et al. (2020) argues that combinatorial thinking is an important mental activity in building one's knowledge or arguments and experiences. It can be concluded that combinatorial thinking is a process of building one's argument and considering all possibilities and combining several objects from the available objects.

Combinatorial thinking is an aspect of students' mathematical thinking that is closely related to solving the problems used (Manohara, et al., 2019). Thamsir et al. (2019) argues that through problem solving skills training, students are trained to think, develop curiosity, and are expected to build confidence in solving problems. Thus, skills in solving problems can improve students' thinking skills. Based on the research results of Eizenberg & Zaslavsky (2004), one of the most difficult mathematical topics to teach and learn is combinatorics. Siddikov Z.Kh. (2022) argues, using combinatoric problems in the assignment and learning of mathematics has an impact on the development of combinatorial thinking. In their research, Eizenberg & Zaslavsky (2004) argue that most combinatoric problems do not have available solution methods, and create a lot of uncertainty about how to approach their solution and what methods will be used to solve the problems. Therefore there are five types of verification strategies used by students in ensuring the correctness of solving combinatoric problems according to Eizenberg & Zaslavsky (2004) as follows.

1. Verification by reworking the solution, problems that have been resolved will be reworked using the same method by viewing and checking all or part of the work a second time.
2. Verification by adding justifications to the solution, adding justification to the settlement solution to support the correctness of the solution.
3. Verification by evaluating the reasonability of the answer, the final results that have been obtained are viewed and tried to be checked for reasonableness either by estimation or checking in general, by comparing the size of the results space.
4. Verification by modifying some components of the solution, changing the representation that has been used in the final solution, or trying to implement the same solution method using a lower number.
5. Verification by using a different solution method and comparing answers, comparing the results of solving the initial method with other different methods. (Eizenberg & Zaslavsky, 2004).

Coenen et al. (2018) conducted research on several children aged 14-16 years, that children with this age range experienced the development of combinatorial thinking. Schools in Indonesia, children with an average age of 16 years are students of class XI at the high school level. Kemendikbudristek (2022) states that one of the achievements of learning mathematics in the independent curriculum in phase F (for class XI and XII SMA) is "*Peserta didik memahami konsep peluang bersyarat dan kejadian yang saling bebas menggunakan konsep permutasi dan kombinasi*". The results of Coenen's research (2018) show that the peak period for the development of combinatorial thinking is in children aged 16 years, in this case, high school class XI students. In addition, Medová et al. (2020) also argue that the focus on combinatorics should be given to high school students. Thus, this research examines the combinatorial thinking processes of 16-year-old high school students.

Lockwood (2012) states that there is a combinatorial thinking model which consists of three components and are connected to one another as shown in Figure 1 below.



(Lockwood, 2012)

Figure 1. Combinatorial Thinking Models

Formulas/expressions refer to the use of formulas and cases that produce numeric values. Counting processes refer to a series of calculation processes in which counters are involved in calculating and or solving problems. Sets of outcomes refer to a collection of object results that have been calculated (Lockwood, 2012).

In one unit of the combinatorial thinking model by Lockwood (2012) above, Rezaie & Gooya (2011) put forward four levels of combinatorial thinking as follows.

1. Level 1 (Investigating "Some Cases"), students investigate several existing cases. This level occurs in the Formulas/Expressions component in terms of investigating all cases of the problem, and mathematical statements (Rezaie & Gooya, 2011).
2. Level 2 ("How am I sure that I have counted all the cases?"), students convince the truth of solving problems in their own way. Usually through the Systematic Listing method disclosed by Lockwood & Gibson (2014).
3. Level 3 (Systematically Generating All Cases), related to the process students carry out in concluding and generalizing problems. Generally this level is found in patterns aiming at Formulas/Expressions (Rezaie & Gooya, 2011).
4. Level 4 (Changing the Problem into another Combinatorial Problem), students are asked to validate the conclusions generated by working on different problems but the same in the context of their completion. Generally, models of combinatorial thinking at this level come from formulas/expressions (Rezaie & Gooya, 2011).

From the description above, combinatorial thinking seen from the level of combinatorial thinking can be seen in table 1 as follows.

Table 1. Combinatorial Thinking Level Indicator

Level	Combinatorial Thinking Level	Indicator
1	Investigating "Some Cases"	Able to explain the intent and purpose of the problem.
		Able to determine the cases or information from the problem.
2	"How am I sure that I have counted all the cases?"	Visualizing alternative answers to the given problem, one of which is by means of Systematic Listing (listing all possible combinations of available objects) or other strategies.
3	Systematically Generating All Cases	Able to solve the first problem.
		Able to draw conclusions from the whole answer.
		Able to discover new concepts.

Level	Combinatorial Thinking Level	Indicator
4	Changing the Problem into another Combinatorial Problem	Able to solve the second problem.
		Give examples of other similar combinatorics problems.
		Apply similar solving concepts to other combinatoric problems that have already been encountered.

(Adaptation from Rezaie & Gooya, 2011)

METHOD

This research uses a qualitative approach. This type of qualitative research is descriptive and uses analysis (Moleong, 2012). This is consistent with the purpose of this research, which is to describe combinatorial thinking in high school students in solving combinatoric problems. To get research paticipants, a Mathematics Ability Test (MAT) was administered. Data were obtained in the form of MAT scores and categorized into three group categories namely high, medium and low groups. The MAT referred to in this research consists of three probability questions. Table 2 below shows the formula for categorizing mathematical abilities.

Table 2. Mathematical Ability Categorization Formula

Student MAT Score	MAT Category
$88 \leq x \leq 100$	High
$60 \leq x < 88$	Medium
$x < 60$	Low

(Adaptation from the Kemendikbud, 2017)

The research participants that have been obtained will complete the Combinatoric Problem Task (CPT) which consists of one problem regarding determining the number of ways and opportunities. The following is the combinatorics problem listed on the CPT.

Ari has a safe containing his savings, but Ari forgets the PIN combination for the safe. The safe is automatically blocked if an incorrect PIN input occurs at most three times. Ari remembered that the safe's PIN consisted of a 4-digit combination and did not contain the numbers "3", "4", "7" and "9". The following figure shows the sequence of the number combinations on the safe PIN.

1st digit	2nd digit	3rd digit	4th digit

In addition, Ari remembers several things that,

- The combination of these numbers forms thousands which are more than 3000 and are even numbers
- The 1st and 4th digits must be the same
- The 2nd number is different from the 3rd number
- The 3rd number is different from the 4th number

a. What is the probability that Ari can open the safe on the first try?
b. If Ari has failed to open two times, what is the probability that he succeeded on the last try?

Figure 2. Combinatorics Problem Task (CPT)

After the research participants solved the problems in the CPT above, then a task-based interview was conducted to find out the students' combinatorial thinking processes. The data obtained was then analyzed based on Miles & Huberman (1992), namely by reducing data, presenting data, and drawing conclusions. The results of the analysis are in the form of a description of combinatorial thinking seen from the combinatorial thinking model, the level of combinatorial thinking, and the strategy for verifying combinatoric problems.

Combinatorial thinking in this research was analyzed based on the thinking processes of high school students in solving combinatoric problems, through the combinatorial thinking model initiated by Lockwood(2012). Each process in the thinking model is analyzed and described the level of combinatorial thinking used by students according to Rezaie & Gooya (2011). Also in solving combinatoric problems, students use different verification strategies for each student (Eizenberg & Zaslavsky, 2004). Finally, a description of the combinatorial thinking in this research was obtained.

RESULTS AND DISCUSSION

The process of finding research participants was carried out at SMA Negeri 1 Sidayu on 8th of May 2023. Students of XI-1 class were given MAT and obtained student MAT scores and grouped in Table 3 below.

Table 3. Table of Distribution of Student MAT Score Groups

Student MAT Score	Many Students		MAT Category
	Male	Female	
$88 \leq x \leq 100$	2	0	High
$60 \leq x < 88$	3	1	Medium
$x < 60$	11	12	Low

Table 4. Selected Research Participants

Student's Initial Name	Research Participant Code	Gender (M/F)	MAT score	MAT Category
MKR	S_H	M	98	High
AFT	S_M	M	71	Medium
AMDF	S_L	M	32	Low

From Table 3, it shows that there are no female students who are candidates for research participants with high mathematical abilities, so that the determination of the research participant will be taken from 16-year-old male students in the mid score category for each mathematical ability group. All selected research participants are of the same gender, this aims to be equal, so that research only focuses on mathematical abilities. Table 4 shows the students who were selected as research participants along with the participant code. The selection of research participants was also approved by the partner teacher in the field of mathematics with various considerations such as the ability to communicate so that the interview process ran smoothly.

The three selected research participants were given CPT to work on and interviews were conducted based on the results of student work. The combinatorial thinking process of high school students in solving combinatoric problems is obtained as follows.

Combinatorial Thinking High School Students with High Mathematical Ability

The following is the process of S_T in solving combinatoric problems sequentially based on the components of the combinatorial thinking model.

Formulas/Expressions → Counting Processes

In the task-based interview session, research participant S_H provides a review of the aims, objectives, and case identification shown in the transcript below.

P : What is the meaning and purpose of this problem?
S_H : To determine the correct chance of finding the safe code that him forgot, right?
P : Right, then what's your idea in answering the first question?

S_H : *What I know is, if we determine an opportunity, we have to have a sample, right? So, the sample here is not yet known, that is, there are many possible arrangements of numbers that are correct. So I'll determine the many possible PINs first,*

P : *Oh, is that so?, then how do you determine the possibility of the PIN?*

S_H : *Let me see first, this is known in the problem so the possible numbers are "0", "1", "2", "5", "6", and "8", then it is an even number of more than 3000. But the condition is the first number is the same as the last number, even and more than 3, meaning that if not "6 with 6" yes "8 with 8". The second and third numbers are free, as long as they are different from the others, so these are the remaining numbers that have been used.*

From the interview transcript above related to the Formulas/Expressions component above, S_H understood the intent and purpose of the problem given, namely to determine the probability of finding a forgotten safe PIN. In addition, S_H also identify the cases that are in the problem. Identification of cases carried out by S_H is in the form of images with implied meaning as shown in Figure 3 below.

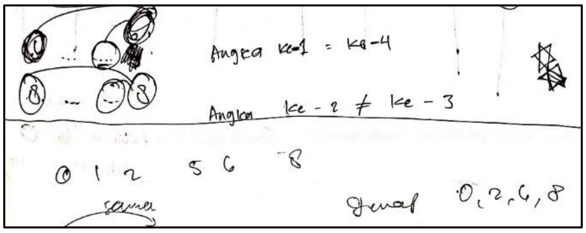


Figure 3. Case Identification by S_H

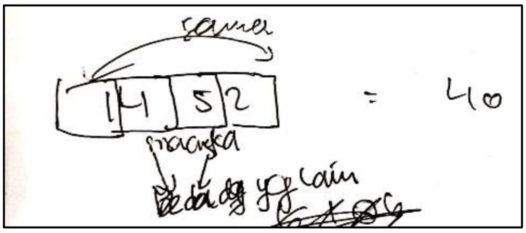


Figure 4. Formula used S_H

From Figure 3 above, S_H identifies the cases shown in the upper left schematic. After disclosing and investigating the case, S_H then applies the multiplication rule as shown in Figure 4 above. To find out S_H's thought process in using formulas related to the calculation process, the researcher gave questions in the interview session as in the transcript below.

S_H : *… means that maybe if not "6 with 6" then "8 with 8". The second and third numbers are free, as long as they are different from the others, so these are the remaining numbers that have been used. Then I multiply all to find 40*

P : *Where do you know how to multiply it?*

S_H : *I've never encountered a problem like this*

The transcript above shows the S_H process in using the multiplication rule to determine the many possibilities using the cases that have been disclosed. The multiplication rule becomes the formula that will be used to determine the number of possibilities. The formula that has been determined by S_H is then calculated through counting processes and many possibilities are obtained. In this process, the research participant fulfills the 1st level indicators of combinatorial thinking.

Counting Processes → Sets of Outcomes

After the S_T determines the formula to be applied, then a calculation process is carried out to obtain many possible safe PINs. The Counting Processes carried out by S_H are shown in the interview excerpt below.

S_H : *... so these are the leftover numbers that have already been used. Then I multiply all to find 40*

Based on the transcript above, in this process S_H performs a calculation process from a predetermined formula. In the calculation process, the outcomes are 40 possibilities.

Sets of Outcomes → Expressions

As many as 40 possibilities obtained by S_H using the multiplication rule are not fully believed to be true. Therefore, S_H returns to understand the cases that have been disclosed. The following interview excerpts show that the S_H understands the case again.

P : What is your strategy in ensuring the correctness of solving this problem?

S_H : If to be sure, I understand the conditions then I list all the possibilities from the information above, now I get 40 possibilities, so I'm sure the calculations I calculated earlier are correct

Based on the interview transcript above, in this process S_H determines the sets of outcomes obtained through the multiplication rule. Therefore, S_H investigated the case again at CPT.

Expressions → Counting Processes

It is not enough to reunderstand the cases that have been uncovered, S_H then shows the possible combinations. This can be shown in Figure 5 which shows the possible safe PIN arrangement by S_H.

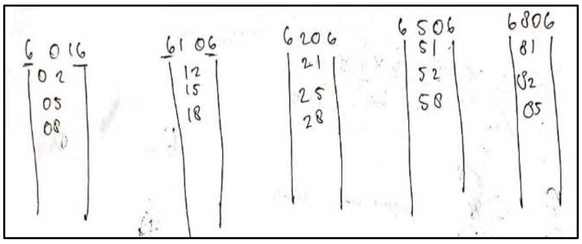


Figure 5. Possible Safe PIN Arrangements by S_H

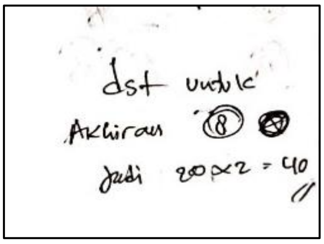


Figure 6. Form of Generalization by S_H

Based on the results of the research participant's work in Figure 5 above, in this calculation process the research participant lists possible PINs based on predetermined cases. How to register all the PIN arrangements is called the systematic listing method. By using this method, the research participant can determine the correctness of the previous answer using the multiplication rule.

P : Oh, so you list 40 one by one and the answers are the same as before? Are you sure you haven't missed some PIN combination?

S_H : Sure, because I put it in order, I repeated the answer the same

From the interview transcript above, the research participant is confident about the results of his answers which state that there are many possible PINs that can be made. Because S_H can be sure of the many possibilities that are obtained. So, S_H in this process fulfills the 2nd level indicator of combinatorial thinking. So, S_H received a possible arrangement of a PIN with the first number and the fourth number with the number "6" only as many as 20 possibilities. Furthermore, S_H gives a statement like in Figure 6 above which shows the form of generalizing S_H by concluding cases where the first and fourth numbers "6" and "8" are the same. So, S_H in this process meets the indicators at the 3rd level of combinatorial thinking, namely systematically generating all cases.

In ensuring the correctness of the answers that have been obtained using the multiplication rule, S_H uses two ways to ensure that the answer is correct. This can be called a verification strategy (Eizenberg & Zaslavsky, 2004). Therefore, it can be concluded that the verification strategy used by S_H in solving combinatoric problems is described as follows.

1. Verification by reworking the solution, the research participant reworks and solves the problem in exactly the same way to check the correctness step by step, and
2. Verification by using a different solution method and comparing answer. The different method used by S_H is the systematic listing method and the multiplication rule. Both methods obtain the same results, so that the research participant concludes that the solution is correct.

Since the research participant can confirm the correctness of the answer, in this process S_H fulfills the 2nd level indicator of combinatorial thinking.

Counting Processes → Sets of Outcomes

Through the calculation process described above, it was concluded that SH got as many as 40 possible safe PIN combinations through the systematic listing method as initiated by Lockwood & Gibson (2014). Many of these possibilities have been confirmed by S_H because they have carried out the various verification strategies described above.

Sets of Outcomes → Counting Processes

To answer the first question, it is necessary to repeat the calculation process, so that we get an answer, namely the probability of determining the safe PIN correctly on the first try.

P : Then how can you answer the first question, namely $\frac{1}{40}$*?*

S_H : There is only 1 PIN that is certain to be correct out of these 40 possibilities, so the answer is yes, $\frac{1}{40}$

P : Okay. Then how do you conclude in solving the problem from the beginning to determine the possibility to determine the opportunity?

SH: So to determine the probability, just multiply each possibility and then the final answer from the first question $\frac{1}{40}$

From the interview transcript above, the research participant explained the calculation process to answer the first question. In addition, S_H also draws a conclusion regarding the overall settlement process. Then S_H fulfills all the indicators at the 3rd level of combinatorial thinking.

Counting Processes → Sets of Outcomes

After carrying out the calculation process as described above, the final result of the S_H's answer to the first question is shown in Figure 7 below.

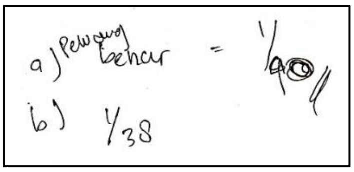


Figure 7. S_H's Final Answers

The research participant determines the outcomes correctly, namely the correct probability of opening the safe on the first try is $\frac{1}{40}$. Furthermore, with the same concept, students answer the second question, namely the correct probability of opening the safe on the last trial is $\frac{1}{38}$.

P : Okay, then the answer to the second question is how come $\frac{1}{38}$*?*

S_H : *What I understand means that the first attempt failed, the second failed, and the third was correct. That means you've probably lost two*
P : Oh the 40 possibilities just disappeared like that?
S_H : *Yes, there are 38 left and 1 is correct. So the chances are* $\frac{1}{38}$

The interview transcript above shows the process of calculating the second question. Furthermore, to find out whether the research participant can fulfill the 4th level of combinatorial thinking, the researcher gives questions related to the application of the concept to more complex similar problems as follows.

P : Try to give examples of problems that are similar to this problem!
S_H : *There are so many, for example the chance to get a beautiful license plate on a condition that all beautiful numbers are the same*
P : If such a problem, can you solve it? How?
S_H : *InsyaAllah ican, in the same way, multiplying to find the possibilities*

From the transcript above, it shows that S_H is able to change combinatoric problems into other, more complex combinatoric problems, and apply the same concept, namely the multiplication rule which becomes a formula for other combinatoric problems. Therefore it can be concluded that in this process, S_H achieves all indicators of combinatorial thinking at the 4th level.

The description above shows the process of S_H in solving combinatorics problems. In general, this process can form a combinatorial thinking model in Figure 8 as follows.

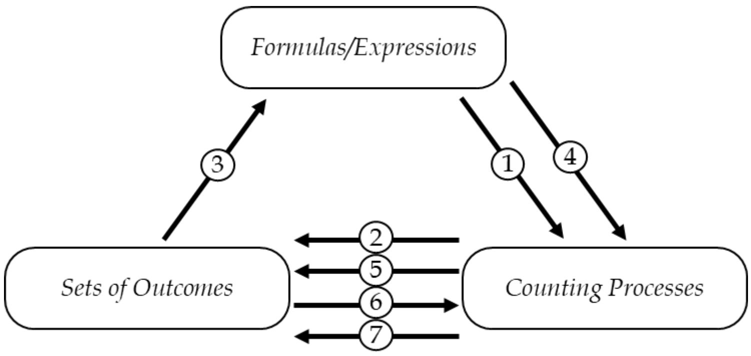


Figure 8. Combinatorial Thinking Model Chart of S_H

The chart above shows the three components of combinatorial thinking which are related to each other cyclically and connected by numbered arrows. The numbers on the arrows are the process sequence of S_H's combinatorial thinking model in solving combinatoric problems in this research. Thus, S_H applies the same combinatorial thinking model in solving other similar problems, namely *Formulas/Expressions → Counting Processes → Sets of Outcomes → Expressions → Counting Processes → Sets of Outcomes → Counting Processes → Sets of Outcomes .*

Combinatorial Thinking of Students with Medium Mathematical Ability

The following is the process of S_M in solving combinatoric problems sequentially based on the components of the combinatorial thinking model.

Formulas/Expressions → Counting Processes

In the early stages of solving the problem, S_M understands the intent and purpose of the problem given. Through interviews, S_M provides a review of the aims, objectives, and case identification of the problems shown below.

P : Explain what you know of the intent and purpose of this problem?
S_M : To determine the probability of successfully opening a safe that forgets the PIN
P : How do you think about solving this problem?
S_M : I'm looking for the total possible PIN first
P : What for? previously you thought that the end goal is to determine opportunities, then why are you looking for possible PINs in the first place?
S_M : To calculate the probabilities later, to look for opportunities you need a sample space, that is, there are many possible PINs

Based on the results of the transcript above, in this process S_H understands the intent and purpose of the given problem and identifies the cases in the problem. Figure 8 below shows the results of case identification by S_M .

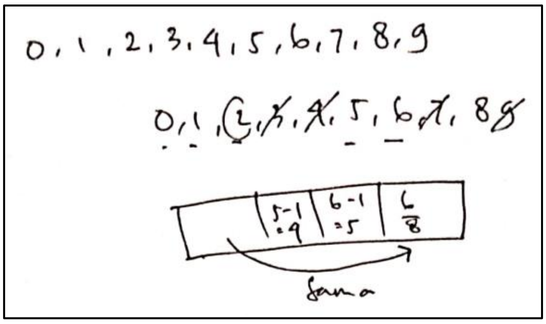


Figure 9. Case Identification by S_M

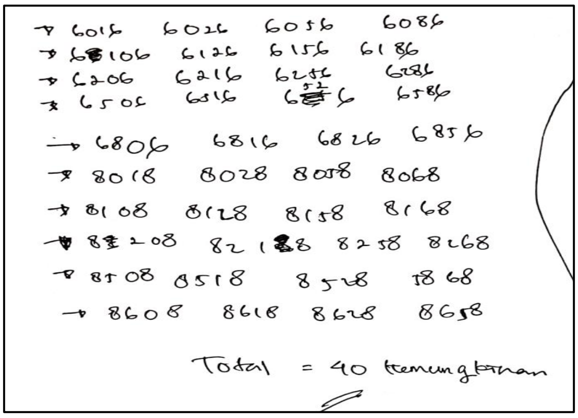


Figure 10. Arrangement of Possible Obtained S_M

From Figure 9 above, it can be seen that the determination and investigation of 'cases' is used by S_M . These cases are expressions components in the combinatorial thinking model. Through the cases that have been put forward by S_M, then the research participant determines the many possible PINs that can be arranged. To find out the next settlement process, interviews were conducted as follows.

P : Okay, then how do you find the total possibilities?
S_M : Initially I specified the possible numbers in the first, second, third, and fourth digits. Then I search one by one in order like this, now I will get 40 possibilities

From the interview transcript above, it shows that S_M has found 40 possibilities and is presented on the work sheet as follows. Figure 10 above shows that S_M lists all the PIN combinations sequentially. The research participant obtains the results of 40 possible PINs which can be referred to as sets of outcomes. Arranging all of these possibilities is one way for the research participant to ensure the correctness of the answers. This method is called Systematic Listing (Lockwood & Gibson, 2014).

P : How can you be sure there are only 40 possibilities in total? Are you sure you haven't missed any PIN?
S_M : Sure sis. The problem is that I have sorted it, starting from 6016, 6026, 6056, 6086, ... etc.

From Figure 10 and the interview transcript above, it can be concluded that the systematic listing pattern used by S_H with medium mathematical abilities is to sort combinations of numbers starting from the smallest number to the largest number and list 40 possibilities in full. By sequencing all of these combinations, S_M was sure not to miss the safe PIN number combination. Therefore S_M at this stage fulfills the indicators of the 1st and 2nd levels of combinatorial thinking.

Sets of Outcomes → Formulas/Expressions

To strengthen the argument that there are 40 possible PINs as sets of outcomes. Figure 11 below shows that the research participant applies another way, namely the multiplication rule in solving problems.

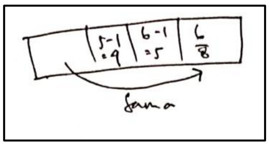


Figure 11. *S_S in Using Other Methods*

P : If you use this method, are you sure that this method is correct?
S_H : I also used another method, sis, but I don't know if it's right or wrong, but I found the same answer, namely 40. So I'm sure it's true

From Figure 11 and the interview transcript above, it shows that S_M has never used the multiplication rule, so S_M has doubts about this method. The research participant is not sure whether it is right or wrong, but because he gets the same answer, S_M concludes that the result of the solution is correct. At this stage, the verification strategy used by S_M appears, namely verification by using a different solution method and comparing answers. The research participant uses the systematic listing method and the multiplication rule which has the exact same answer, so that the research participant feels confident that the answer can be said to be correct.

Formulas/Expressions → Counting Processes

The formula that has been determined by the research participant is then carried out by the calculation process. The following is an excerpt from the interview transcript of S_M regarding the process of using and calculating the multiplication rule as a formula.

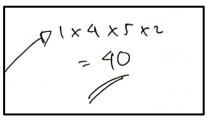


Figure 12. S_M Answer Using Multiplication Rule

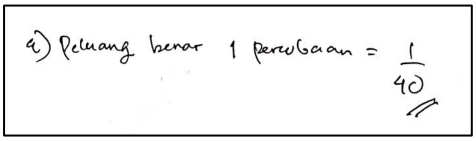


Figure 13. S_M Probability Answer

S_M : Yes, the result is 5, now the second digit is also the remainder, which is 4 possibilities, from 5 minus 1, the result is 4. Then I multiply this $1 \times 4 \times 5 \times 2 = 40$ probabilities. (Figure 1)
Because both are 40, so I'm sure it's true.
P : Oh, then how do you think about answering the first question?
SM : $\frac{1}{40}$, this. (Figure 13)
P : Oh, that's the final answer?. So, what conclusions can be drawn from solving this problem? How's the chance $\frac{1}{40}$?
SM : Because only 1 out of 40 possibilities is correct.

Based on the transcript above, in this process the research participant uses a predetermined formula for the calculation process. On the other hand, S_M answered the first question correctly and drew a conclusion. Then S_M fulfills all the indicators at the 3rd level of combinatorial thinking. After the research participant is sure that there are 40 possible safe PIN arrangements, then the research participant completes the first problem.

Counting Processes → Sets of Outcomes

After going through the process of calculating the formula, namely the multiplication rule. The research participant determined the set of results which stated the number of possible safe PINs, namely there were 40 possibilities. Of the 40 possibilities listed, one of them is the correct PIN. Then S_M concludes that the probability of being right in the first experiment is $\frac{1}{40}$.

Sets of Outcomes → Counting Processes

In the next stage, many possible PINs will be calculated to answer the second question.

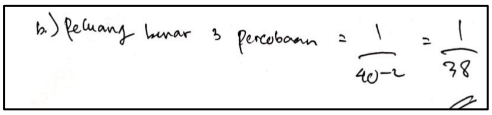


Figure 14. SM Answer for the 2nd Question

P : How can the answer to this second question $\frac{1}{38}$? Explain!
S_M : It's almost the same way, because from $\frac{1}{40}$ earlier it was subtracted by 2, because it was for the first and second experiments, so it resulted in $\frac{1}{38}$ what was left like this. (Figure 14)
P : So 1 in 38 is like that.

From the interview transcript above, it can be seen that S_M carried out the calculation process by means of the total probability minus the number of trials, so that 40 was reduced by 2 trials. Therefore, the correct PIN code is 1 PIN among the remaining 38 possible PINs.

Counting Processes → Sets of Outcomes

In the next stage, the results of the final answer to the second question will be obtained, namely $\frac{1}{38}$. Furthermore, to find out whether S_M fulfills the further fourth level indicators or not, the researcher asks the following questions.

P : Can you provide an example of a problem similar to this one? Can you find other examples?
S_M : Yes I can, for example, if we forgot our luggage password
P : If the problem is as you envision it, how do you determine the number of possibilities?

S_M : I made a list like before
P : But what if there are hundreds or even thousands? Do you still use list method?
S_M : Oh, if that's the case, I'll use the one multiplication

From the interview transcript above, it was found that S_M answered the second question correctly, gave examples of similar combinatoric problems, and applied a new concept that had been set before, namely the multiplication rule. Therefore, the medium research participant fulfills all the indicators at the 4th level of combinatorial thinking.

The description above shows the process of S_M in solving combinatoric problems. In general, this process can form a combinatorial thinking model in Figure 15 as follows.

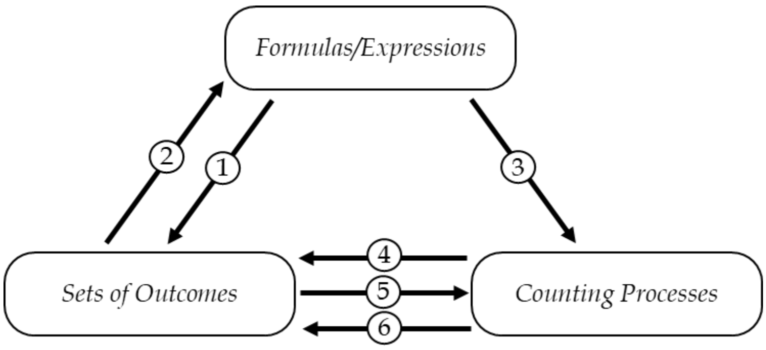


Figure 15. Combinatorial Thinking Model Chart of S_M

Thus, S_M applies the same combinatorial thinking model in solving another similar problem, namely *Expressions → Sets of Outcomes. → Formulas → Counting Processes → Sets of Outcomes → Counting Processes → Sets of Outcomes* .

Combinatorial Thinking of Students with Low Mathematical Ability

The following is the S_L process in solving combinatoric problems sequentially based on the components of the combinatorial thinking model.

Expressions → Sets of Outcomes

In the early stages of solving the problem, S_L understands the intent and purpose of the given problem. Through the interview session, S_L provides a review of the aims, objectives, and case identification of the problems shown in the interview transcript as follows.

P : Can you understand the meaning of this question? Try to explain!
S_L : Ari couldn't open the safe because he forgot his PIN, so we were told to look for opportunities for Ari to open the safe.
P : How is your idea in determining the opportunity?
S_L : I'm looking for a possible PIN first, because we already know the cases.
P : What are the cases, say it!
S_L : Does not contain the numbers "3", "4", "7", and "9", which are thousands and even numbers, the 1st number = the 4th number, which is an even number and must be more than "3", means what can numbers "6" and "8". 1st number $\neq$ 2nd number $\neq$ 3rd number.

The transcript above shows that in this process S_L understands the intent and purpose of the given problem and identifies the cases in the problem. Case investigations that use low-ability research participants are also presented in the S_L on the work sheet as shown in the following figure.

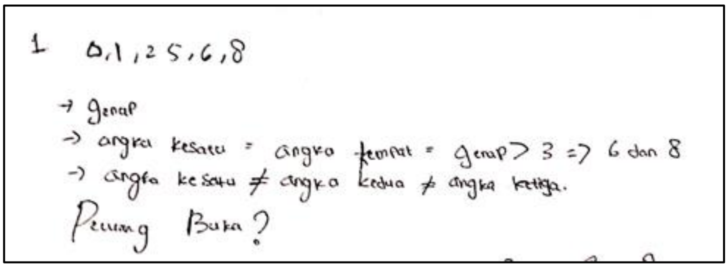


Figure 16. Case Identification by S_L

From Figure 16 above, the case investigation carried out by S_L only copied the same sentence as on the CPT sheet. The cases in CPT are not reviewed, so that low-ability research participant starts the problem-solving process by stating expressions in the components of the combinatorial thinking model. Through the expressions that have been stated by S_L , a safe PIN combination arrangement can be made using the systematic listing method. Then a set of results is obtained which becomes the sets of outcomes in Figure 17 below.

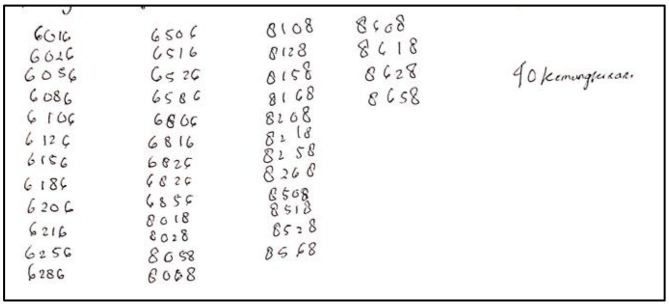


Figure 17. Safe PIN Combination Arrangement by S_L

Figure 17 above shows the systematic listing process used by S_L, namely the research participant sorts from the smallest number to the largest number and lists as many as 40 possibilities in full but are not grouped according to their respective criteria. The use of this method can ensure confidence in the correctness of answers (Lockwood & Gibson, 2014). The transcript below shows the research participant's confidence in S_R in determining the possibility.

P : Certain?
S_H : Wait a minute, I'll count it first. It is true that 40
P : So from this visualization process you can be sure that there are 40 possibilities, right?
S_H : Yes InsyaAllah
P : If there are several possibilities that are missed, how do you write them down?
S_H : Mmm.., I don't know, I was just trying to make a list, I don't know if it's true or not

Based on the interview transcript above, because S_L registered by not grouping every possibility that existed, S_L who was initially sure became doubtful about the results of his work. S_L at this stage fulfills the 1st and 2nd level indicators of combinatorial thinking.

Sets of Outcomes → Counting Processes

Figure 14 above shows that there are 40 possible PINs that can be arranged by S_L through the systematic listing method as a form of component sets of outcomes. Next, a calculation process will be carried out to determine the opportunity to open the safe PIN. The following is the transcript of the $_{SR}$ interview regarding the calculation process in solving the problem.

P : Is there another way of determining this possibility?
S_L : As long as I find questions like this, I always use this list method, I don't understand that any formula
P : Okay, let's say 40 possibilities are true, then how do you answer the first question?
S_L : Right out of these 40 possibilities, only 1 PIN can open the safe, so there's 1/40 possibility

From the interview transcript above, S_L did not use any verification strategy, so that the low-ability research participant was not fully convinced of the Sets of Outcomes which stated 40 possible PINs were composed. Furthermore, with the calculation process, the answers to the first question are obtained, namely the probability of being correct on the first attempt to open the safe. The process of determining these opportunities is included in the Counting Processes component. S_L did not use other methods so that S_L fulfilled most of the indicators at the 3rd level of combinatorial thinking.

Counting Processes → Sets of Outcomes

After going through the Counting Processes stage, S_L obtained the answers (Sets of Outcomes) on the first question correctly. Furthermore, S_L answered the second question correctly. To find out the calculation process, the researchers asked in the interview session. The interview transcript below shows the S_L calculation process in obtaining answers to the second question.

P : Ohh, then how can you answer the second question, $\frac{1}{38}$?
S_L : Like this, in my opinion, the first PIN has failed, the second one has also failed, meaning the probability is reduced by 2. So 40 possibilities are reduced by the 2 that failed earlier to 38, so the chances are that, $\frac{1}{38}$
P : Try to give examples of problems that are similar to this problem!
S_L : Sometimes, I forget my cellphone password, we can use that method
P : If, there are maybe hundreds or even thousands, how do you determine the number of cellphone passwords?
S_L : Mmm, oh yes. If there are lot of numbers needed, I can't list a lot of them, it waste my time.

The interview excerpt above shows the counting processes S_L in answering the second question. To find out whether the research participant can solve other combinatoric problems, the researcher gives questions related to other, more complex combinatoric problems. From the interview excerpt above, it shows that the research participant gave an example of a more complex similar problem, namely a combination of numbers made on a cellphone password. According to S_L in solving the new problem using a systematic listing method like the previous problem. However, the research participant acknowledged that this method could not be used forever. The reason is that the method can only be used on simpler problems with not too many possible arrays. Therefore, S_L fulfills some of the indicators at the 4th level of combinatorial thinking.

The description above shows the process of S_L in solving combinatorics problems. In general, this process can form a combinatorial thinking model in Figure 18 as follows.

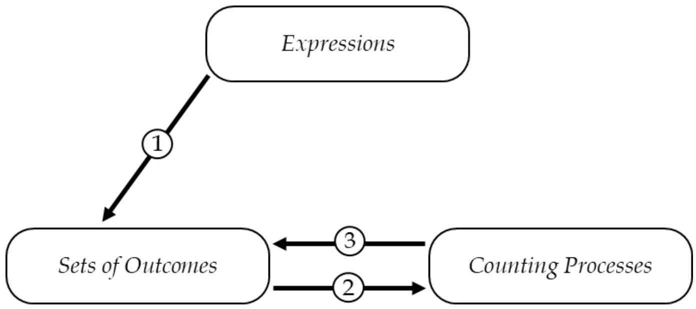


Figure 18. Combinatorial Thinking Model Chart of S_L

Thus, S_L applies the same combinatorial thinking model in solving another similar problem, namely *Expressions → Sets of Outcomes → Counting Processes → Sets of Outcomes* .

Table 5 below shows a comparison table for high school students' combinatorial thinking between mathematical abilities.

Table 5. Table of Comparison Between Categories of Student Ability

	Mathematics Ability Students		
	High	**Medium**	**Low**
Combinatorial Thinking Models	*Formulas/Expressions → Counting Processes → Sets of Outcomes → Expressions → Counting Processes → Sets of Outcomes → Counting Processes → Sets of Outcomes*	*Expressions → Sets of Outcomes → Formulas → Counting Processes → Sets of Outcomes → Counting Processes → Sets of Outcomes*	*Expressions → Sets of Outcomes → Counting processes → Sets of Outcomes*
Level 1 *Investigating "some cases"*	Identify implicit cases in the problem and determine the solution ideas to be used		
Level 2 *"How am I sure that I have counted all the cases"*	▪ Using the *systematic listing method* ▪ Using two types of verification strategies (rework and two different settlement methods)	▪ Using the *systematic listing method* ▪ Using one type of verification strategy (two different settlement methods)	▪ Using the *systematic listing method*
Level 3 *Systematically generating all cases*	▪ Drawing a final conclusion ▪ Finding new concepts that can be applied to other, more complex problems	▪ Drawing a final conclusion ▪ Finding new concepts that can be applied to other, more complex problems	▪ Draw a final conclusion
Level 4 *Changing the problem into another combinatorial problem*	▪ Give examples of other combinatorial problems ▪ Applying new concepts to other, more complex problems	▪ Give examples of other combinatorial problems ▪ Applying new concepts to other, more complex problems	▪ Give examples of other combinatorial problems
Verification Strategy	▪ Verification by reworking the solution ▪ Verification by using a different solution method and comparing answers	Verification by using a different solution method and comparing answers	-

From the table above, even though students with high-medium-low math abilities at level 1 think combinatorially are fulfilled, there are differences between them. Students with high abilities and are currently mentioning and describing existing information, implicitly according to their respective meanings. This is in line with research conducted by Putri

(2022) that students with high and medium abilities can investigate cases and solve problems by finding several alternative solutions. This is supported by Manohara (2019), the students with high and medium ability can mention all the information that can be used to solve problems. And low ability students just rewrite without reviewing the meaning of the information provided.

The systematic listing pattern used by the three students was different. The three students sort from the smallest number to the largest number. High-ability students only list half of the possible possibilities, because the research participant thinks that between the prefix and the suffix "6" and "8" have the same number of possibilities, so it is multiplied by 2. Students who are capable are grouping possible arrangements based on the cases made in systematic listing. Low ability students only arrange in sequence without grouping. This is in accordance with research conducted by Hidayati et al. (2020), the students ascertain truth by using known simple methods, such as systematic listing and or tree diagrams. In the systematic listing method , students are required to visualize the entire set of results which are referred to as sets of outcomes (Lockwood & Gibson , 2014). In line with the research conducted by Uripno & Rosyidi (2019) that students tend to use the sets of outcomes components as checking and reassuring problems that have been resolved. However, the systematic listing method is not fully applicable to every problem, only simple problems. This is in line with previous research, that a series of results is used to ensure correctness and can be used as a solution to simple problems (Rapanca, Wibowo, & Sapti, 2020).

The verification strategy used for high and medium ability students both uses the "Verification by using a different solution method and comparing answers" type. The two methods used are the same, namely the systematic listing method and the multiplication rule (new concept). But there are differences in the order of completion between them. The new concept used between high and medium ability students are also different. Students with high ability use new concepts early in the completion process, so that these new concepts become the main method of determining many possibilities. Besides that, in solving other combinatoric problems, high ability students directly use the new concept (formula). This is in line with research conducted by Rapanca, et al. (2020), that the structure of combinatorial thinking starts from determining the formula then goes through a calculation process, known as counting processes. Students with medium abilities find and use new concepts after finding many possibilities, in other words as a form of cross-checking to ensure correctness, so that it is not the main method. Because it is not the main method of solving this problem, so that in solving other combinatoric problems, medium students still prioritize the systematic listing method first. If it is not possible to use this method, then a new method will be used to solve similar, more complex combinatoric problems.

CONCLUSION AND SUGGESTIONS

High school students with high mathematical abilities use the process of combinatorial thinking models in the order Formulas/Expressions → Counting Processes → Sets of Outcomes → Expressions → Counting Processes → Sets of Outcomes → Counting Processes → Sets of Outcomes. High school students with high math abilities achieve all levels of combinatorial thinking. In addition, the verification strategies used in solving combinatoric problems also vary, namely reworking with the same method and comparing the final results with two different methods.

High school students with medium mathematical abilities use the process of combinatorial thinking models in the order Expressions → Sets of Outcomes → Formulas → Counting Processes → Sets of Outcomes → Counting Processes → Sets of Outcomes. High school students with medium math abilities achieve all levels of combinatorial thinking. The verification strategy used by medium group students is to compare the final results of work using two different completion methods.

High school students with low mathematical abilities use the combinatorial thinking process model in the order Expressions → Sets of Outcomes → Counting Processes → Sets of Outcomes. High school students with low mathematical abilities fulfill some of the indicators for the level of combinatorial thinking. At the first level, students can identify problems, find implied meanings from available cases, and determine solutions. The second level is fulfilled because students use the systematic listing method , but are not fully sure about the number of sets of outcomes obtained because they use only one settlement method. At the third level, students can draw a conclusion from the work but cannot find new concepts that can be applied to other similar combinatoric problems. As a result students cannot apply the same solving method at the fourth level of combinatorial thinking, namely solving other, more complex combinatoric problems, but can provide examples of other similar combinatoric problems. On the other hand, high school students with low mathematical abilities do not apply verification strategies to ensure the correctness of students' work results, so students with low abilities cannot be sure of the complete correctness of their work.

In this research, researchers were less responsive and paid less attention to students' answers to be carried out, so that students answered questions on problems with short answers. Therefore, it must be really considered when students answer the problems given. Other researchers are also expected to use more developed instruments, so they can easily identify students' combinatorial thinking. In this research, there is already a relationship between the combinatorial thinking model and the level of combinatorial thinking along with the verification strategies used by students in solving problems. For this reason, future researchers are expected to be able to develop this research by using other variables related to combinatorial thinking.

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