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TAYLOR SERIES OF TRIGONOMETRIC FUNCTION FOR LOWER ORDER VISUALIZATION

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Abstract

About the simplest kind of functions are polynomials. Under addition, multiplication, integration, and differentiation, they are closed. A function can be roughly expressed in terms of polynomials of a certain degree using Taylor series. This provides a clear picture of the function's local behavior. The Taylor series fits the function at the place where it is computed more closely the more terms there are. In various fields of natural and social science, trigonometric functions are used. In this work, we examine the tailored series of trigonometric functions using Wolfram Mathematica. The visualization of nature is also obtained using the same software, with point 0 and up to 5 order for sin, cos, sec, tan, cot, and cosec.
Keywords: Polynomials, Taylor Sereis, Nautral and Social Science, Trigonometric function, Wolffram Methematica etc.

Abstract

About the simplest kind of functions are polynomials. Under addition, multiplication, integration, and differentiation, they are closed. A function can be roughly expressed in terms of polynomials of a certain degree using Taylor series. This provides a clear picture of the function's local behavior. The Taylor series fits the function at the place where it is computed more closely the more terms there are. In various fields of natural and social science, trigonometric functions are used. In this work, we examine the tailored series of trigonometric functions using Wolfram Mathematica. The visualization of nature is also obtained using the same software, with point 0 and up to 5 order for sin, cos, sec, tan, cot, and cosec.

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INTRODUCTION

The mathematical theorem known as Taylor's theorem was first put forth by Brook Taylor in 1715. The kth-order Taylor polynomial, which is the result of Taylor's theorem, is a polynomial of degree k that approximates a differentiable function around a given point. The Taylor polynomial for a smooth function is the truncation of the function's Taylor series at order k. The function's linear approximation is represented by the first-order Taylor polynomial, and the quadratic approximation is frequently represented by the second-order Taylor polynomial. Several variations of Taylor's theorem exist, some of which explicitly estimate the function's approximation error using its Taylor polynomial. Two Taylor volumes, Methodus incrementor directa and inversa, as well as the pivotal Linear Perspective in the history of mathematics, are published in London in 1715.

Around the turn of the 18th century, as differential calculus advanced, calculations with infinite series started to be performed. The "Finite Difference Calculus," which today has a key position in numerical analysis, was founded by Brook Taylor in the early 18th century. Taylor's main idea was to approximate a given function f using polynomials (x). This would relate to each function generally rather than the piecemeal contacts of earlier studies. By carefully examining the results of the Newtonian approach to differential thinking, with which he was well aware, this concept was put to rest. A curve f (x) can be "locally" approximated with its tangent line, according to the fundamental principle of differential calculus.

Using the values of the function's derivatives at a single point, a Taylor series in mathematics represents a function as an infinite sum of terms. In 1715, the English mathematician Brook Taylor formally developed the idea of a Taylor series. The series is also known as a Maclaurin series if the Taylor series is zero-centered. Colin Maclaurin, a Scottish mathematician, used this particular specific example of Taylor series extensively in the 18th century. A finite number of terms from a function's Taylor series are frequently used to approximate a function. The error in this approximation is quantified by Taylor's theorem. A Taylor polynomial is any finite amount of initial terms in a function's Taylor series. If the limit is present, the Taylor series of a function is the limit of its Taylor polynomials. Even if a function's Taylor series converges at every point, the function may not be equal to the series. An analytical function is one that is equal to its Taylor series in an open interval (or a disc in the complex plane).

The Zeno's dilemma is the consequence of the Greek philosopher Zeno's consideration and rejection of the problem of summing an infinite series to produce a finite result. The mathematical meaning of the conundrum was apparently unsolved until it was picked up by Democritus and later Archimedes. Later, Aristotle gave a philosophical solution to the contradiction. An endless number of progressive subdivisions could be carried out to produce a finite outcome using Archimedes' method of exhaustion. A few decades later, Liu Hui independently used a comparable technique (Boyer & Merzbach, 1991). Madhava of Sangamagrama provided the earliest evidence of the application of Taylor series and closely related techniques in the 14th century (Dani, 2012). Up until the 16th century, the Kerala School of astronomy and mathematics further developed his contributions with a number of series expansions and rational approximations. James Gregory, who also worked in this field in the 17th century, wrote a number of Maclaurin series. However, Brook Taylor, after whom the series are currently named, did not finally offer a general method for creating these series for all functions for which they exist until 1715. The Maclaurin series is named after Edinburgh scholar Colin Maclaurin, who in the 18th century published a specific example of the Taylor result.

THEORY STUDY

Taylor Polynomials and Taylor Series History

You will have the chance to build on your prior calculus knowledge with Taylor polynomials, which are a logical extension of linearization. The concept of Taylor polynomials is further developed by Taylor series. The surprising and spectacular ways in which Taylor series connect a variety of mathematical topics adds to their allure.

By 1636, Fermat and Roberval both understood that when n is a positive number and x is measured at sufficiently tiny intervals between 0 and X, the value of $\sum_{x=0}^{X} x^n$ equals about $\frac{X^n}{n+1}$. The quadrature of curves with the form x^n was determined by both parties using this relationship. In 1644, Fermat and Torricelli exchanged letters over the matter. Barrow was the first to understand that the inverse of the problem of quadratures was the problem of tangents, and vice versa. He understood the significance of the relationship, which has since become the basic principle of differential and integral calculus, as well as its generality. The work of Wallis in the Arithmetica infinitorum from 1656 served as the foundation for Mercator's approach. Because his "integration rule" failed for $n = -1$, Wallis, like Fermat before him, was unable to handle the rectangular hyperbola. By switching axes to work with $\frac{1}{1+a}$ rather than $\frac{1}{a}$, Mercator was able to get around the issue. He then used long division to create a power series for $\frac{1}{1+a}$. Using Wallis' rule, he was then able to "integrate" terms one at a time (Kouki & Griffiths, 2015).

Application of Taylor Series

The power flow analysis of electrical power networks uses the Taylor Series (Newton-Raphson method). You can estimate your function as a sequence of linear or quadratic forms and then iterate on them consecutively to get the ideal value using multivariate Taylor series, which can be used in many optimization strategies. If the functional values and derivatives are identified at a single point, the Taylor series is used to calculate the value of the entire function at each point. Numerous mathematical arguments are streamlined by the Taylor series representation. An approximate representation of the entire series can be found in the sum of partial series. Numerous optimization methods can make use of multivariate Taylor series. The power flow analysis of electrical power systems uses this series. As rough approximations of the complete function, the partial sums (the Taylor polynomials) of the series can be employed. If enough terms are included, these approximations are accurate.

Power series differentiation and integration can be done term by term, making it particularly simple. A holomorphic function on an open disk in the complex plane is only ever extended to an analytical function. This makes complicated analytic' machinery available. Numerical computations of function values can be performed using the (truncated) series (often by recasting the polynomial into the Chebyshev form and evaluating it with the Clenshaw algorithm). The power series representation makes algebraic operations simple; for example, the Euler's formula is derived from Taylor series expansions of the trigonometric and exponential functions. In disciplines like harmonic analysis, this finding is fundamentally significant. For a limited domain, approximations utilizing the first few terms of a Taylor series can solve issues that would otherwise be intractable; physics frequently use this technique.

METHOD

Taylor Series and Maclaurin Series

Since the terms get smaller and smaller, we can approximate the original quantity by using only the first few terms of the series. The main goal of series is to describe a given complicated quantity as an infinite sum of simple terms. In this section, we finally create the method for writing any logical function as an explicit power series, which most often enables us to accomplish this. Consider the following power series function(f):

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-4)^4 + \cdots \dots \dots \dots |x-a| < R \quad (1)$$

The coefficients cn must be in terms of f, therefore let's try to figure it out. To start, note that if we enter $x = a$ in equation (1), all terms after it are 0 and we obtain as $f(a) = c_0$. On differentiate equation (1) and solving we get

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots \dots \dots \dots |x-a| < R \quad (2)$$

Considering the case $x = a$ and solving we get from equation (2), $f'(a) = c_1$. Again on differentiating equation (2) both side and solving we get,

$$f''(x) = 2c_2 + 2.3c_3(x-a) + 3.4c_4(x-a)^2 + \cdots \dots \dots \dots |x-a| < R \quad (3)$$

Considering the case $x = a$, in equation (3) and solving we get, $f''(a) = 2c_2$. Again differentiating equation (3) both side and solving we get,

$$f'''(x) = 2.3c_3 + 2.3.4c_4(x-a) + 3.4.5c_5 + \cdots \dots \dots \dots |x-a| < R \quad (4)$$

Considering the case $x = a$, in equation (4) and solving we get, $f''(a) = 2.3c_3 = 3!\,c_3$. On continues the same pattern, for case $x = a$, one can get,

$$f^{(n)}(a) = 2.3.4.\dots\dots\dots\dots.nc_n = n!\,c_n \quad (5)$$

On solving equation (5) for nth coefficient of c_n, we get $c_n = \frac{f^n(a)}{n!}$. If f is represented as an expansion of a power series at a, then

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \text{ and } |x-a| < R \text{ then } c_n = \frac{f^n(a)}{n!} \quad (6)$$

One can see that if f has a power series expansion at a, it must take the following form after substituting this formula for c_n back into the series.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \quad (7)$$

Expanding equation (7) on can expressed as

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}((x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots.. \quad (8)$$

The Taylor series of the function f at an is the name of the series in equation (8) (N.d., 2022).

For the case $a = 0$, the function f can be expressed from equation (8) as,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(x)^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots.. \quad (9)$$

The specific term Maclaurin series is given to this circumstance since it occurs frequently enough. Any terms in a maclaurin series expansion must be nonnegative integer powers of the variable. The Puiseux series and the Laurent series are two furthermore all-encompassing series types.

Taylor series with reminder

f is said to be an C^n function on (a, b) if f nth derivative $f^{(n)}$ is continuous on (a, b) and belong to C^∞ if every differentiable of f exists on (a, b). Let us consider f belongs to C^∞ on $(-R, R)$ then $n \in N$, and $x \in (-R, R)$ and we have Taylor series as

$$f(x) = f(0) + f'(0) + \frac{f''(0)}{2!}x^2 + \cdots \dots \dots . + \frac{f^n(0)}{n!}x^n + R_n(x) \quad (10)$$

The Taylor function f is said analytical at 0 when $R_n(x) \to 0$ as $n \to 0$. In addition, remainder has two forms as shown in equation (11) and (13) as

$$R_n(x) = \frac{1}{n!}\int_0^x f^{(n+1)}(t)\,(x - t)^n dt \qquad (11)$$

On integrating by part equation (11) yield,

$$\frac{1}{n!}\int_0^x f^{(n+1)}(t)\,(x-t)^n dt = -\frac{1}{n!}f^{(n)}(0)x^n + \frac{1}{(n-1)!}\int_0^x f^{(n)}(t)\,(x - t)^{n-1}dt \qquad (12)$$

$$R_n(x) = \frac{1}{(n+1)!}f^{(n+1)}(c)x^{n+1} \quad c \in (0,x) \qquad (13)$$

Taylor Series with different variable

Let f(x) is a function with (n + 1) derivative as $f^{(n+1)}(x)$ and continuous in between(x_L, x_R) as $x_L < x < x_R$ then for a and x belong to (x_L, x_R) can be written as (Holmes, 2009),

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \cdots \ldots + \frac{1}{n!}(x-a)^2 f^{(n)}(a) + R_{n+1} \qquad (14)$$

Where $R_{n+1} = \frac{1}{(n+1)!}(x-a)^{n+1}f^{(n+1)}(\eta)$ is named as remainder with a point η in between a and x. In other hand equation (14) can be written in another form as

$$f(x+h) = f(x) + hf'(x) + \frac{1}{2!}h^2 f''(x) + \cdots \ldots \frac{1}{n!}h^n f^{(n)}(x) + R_{n+1} \qquad (15)$$

Where x and x + h belong to (x_L, x_R). Equation (14) and (15) is the Taylor series with single variable. For two variable Taylor series is expressed as $f(x + h)$ and written as

$$f(x+h, t+k) = f(x,t) + Df'(x,t) + \frac{1}{2!}D^2 f''(x,t) + \cdots \ldots \frac{1}{n!}D^n f^{(n)}(x,t) + R_{n+1} \qquad (16)$$

Where $D = h\frac{\partial}{\partial x} + k\frac{\partial}{\partial t}$ and in other hand the quadratic terms two variable taylor series is expressed as

$$f(x+h, t+k) = f(x,t) + hf_x(x,t) + kf_t(x,t) + \frac{1}{2}h^2 f_{xx}(x,t) + hkf_{xt}(x,t) + \frac{1}{2}k^2 f_{tt}(x,t) + \cdots \ldots \ldots$$

Where $f_{xt} = \frac{\partial^2 f}{\partial x\,\partial t}$ is partial derivative with respect to t and x variable with continuous up through order n + 1, Equation (16) and (17) is the Taylor series with single variable. For multivariable Taylor series is expressed as f(x + h) and written as

$$f(x+h) = f(x) + Df(x) + \frac{1}{2!}D^2 f(x) + \cdots \ldots \frac{1}{n!}D^n f(x) + R_{n+1} \qquad (18)$$

Where $x = (x_1, x_2, \ldots \ldots \ldots x_k), h = (h_1, h_2, \ldots \ldots \ldots h_k)$ and $D = h.\nabla = h_1\frac{\partial}{\partial x_1} + h_2\frac{\partial}{\partial x_2} + \cdots .. + h_k\frac{\partial}{\partial x_k}$.

RESULTS AND DISCUSSION

Now that we have generated a number of Taylor series instances using the definition, we can see that almost all of the labor involved in discovering a Taylor series is spent locating the coefficients. By taking derivatives, this may be accomplished for any function; however, determining the derivatives for some functions can be exceedingly time- and effort-consuming due to their complexity. For function f = cos(x) and its taylor sereis is $1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^7)$ converges when $|x| < 1$. The graphical visulaization of cos(x) taylor sereis with point zero and order 6 is shown in figure 1. While other lines display for n orders, the red line represents the Taylor series' nature with point zero and order 5. The Taylor series of cosx has a maximum positive value recorded of 1 and a range from a negative to a positive value at point 0 and order 5 with symmetric nature.

For function f = sin(x) and its taylor sereis is $x - \frac{x^3}{6} + \frac{x^5}{120} + O(x^7)$ converges when $|x| < 1$. The graphical visulaization of sin(x) taylor sereis with point zero and order 5 as shown in figure 2. While other lines display for n orders, the red line represents the Taylor series' nature with point zero and order 5. The Taylor series of sinx has a maximum positive value recorded of 1 and a range from a negative to a positive value at point 0 and order 5.

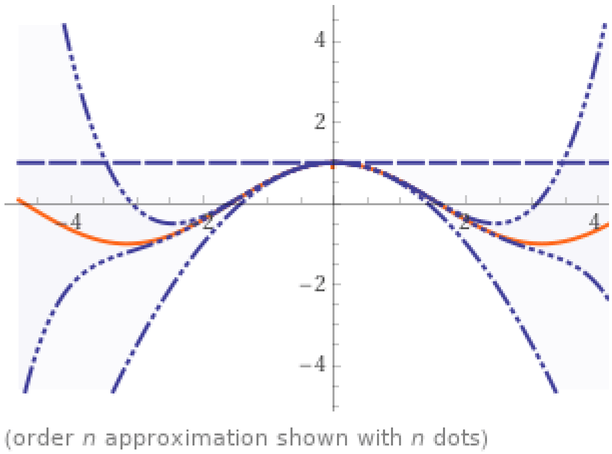


Figure 1: Visualiztion of Taylor sereis of triognmitc fucntion cosx

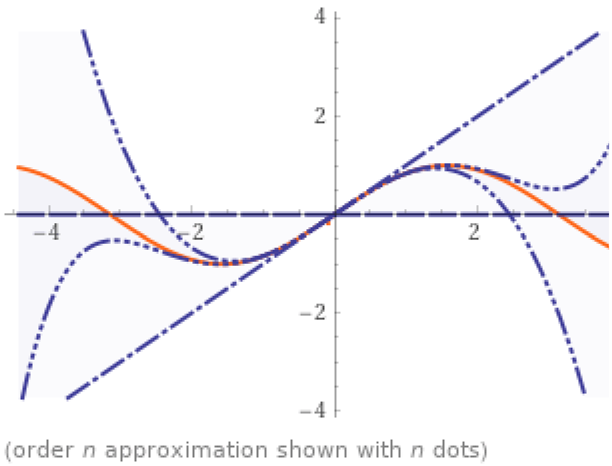


Figure 2: Visualiztion of Taylor sereis of triognmitc fucntion sinx

For function f = tan(x) and its taylor sereis is $x + \frac{x^3}{3} + \frac{2x^5}{15} + O(x^7)$ and the he graphical visulaization of sin(x) taylor sereis with point zero and order 5. While other lines display for n orders, the red line represents the Taylor series' nature with point zero and order 5 as shown in figure 3. The Taylor series of tanx has a maximum positive value recorded up to 4 and a range from a negative to a positive value at point 0 and order 5 with symmetric nature.

For function f = sec(x) and its taylor sereis is $1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + O(x^7)$ and the graphical visulaization of sec(x) taylor sereis with point zero and order 6. While other lines display for n orders, the red line represents the Taylor series' nature with point zero and order 5 as shown in figure 4. The Taylor series of secx has a maximum positive value recorded up to 4 and a range from a negative to a positive value at point 0 and order 5 with symmertic nature.

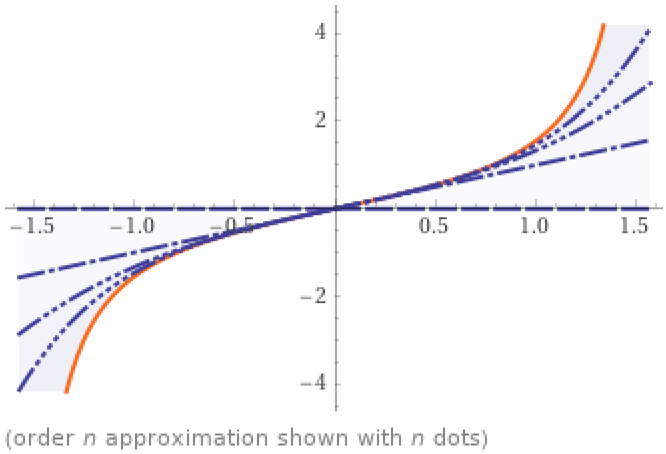


Figure 3: Visualiztion of Taylor sereis of triognmitc fucntion tanx

For function f = sec(x) and its taylor sereis is $1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + O(x^7)$ and the graphical visulaization of sec(x) taylor sereis with point zero and order 6. While other lines display for n orders, the red line represents the Taylor series' nature with point zero and order 5 as shown in figure 4. The Taylor series of secx has a maximum positive value recorded up to 4 and a range from a negative to a positive value at point 0 and order 5 with symmertic nature.

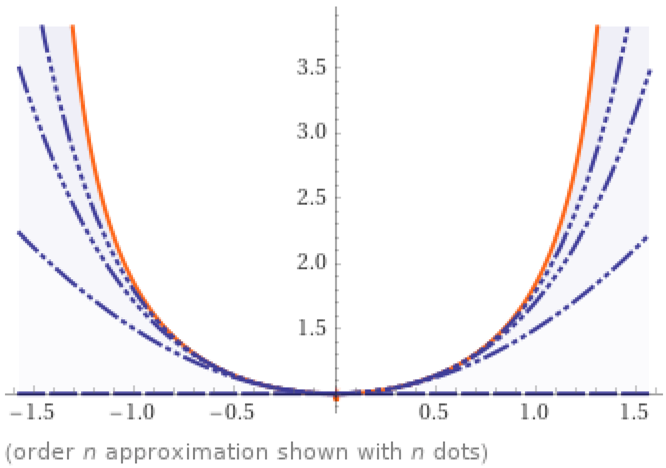
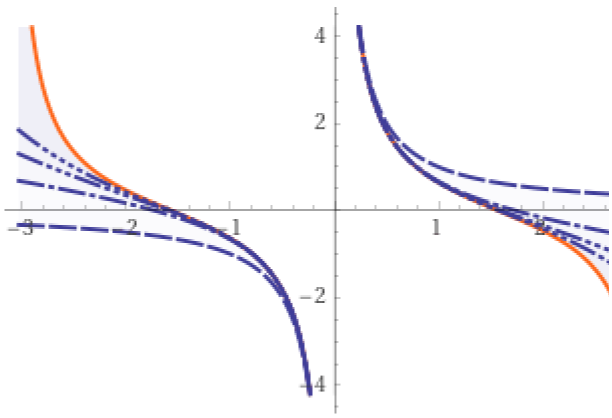


Figure 4: Visualiztion of Taylor sereis of triognmitc fucntion secx

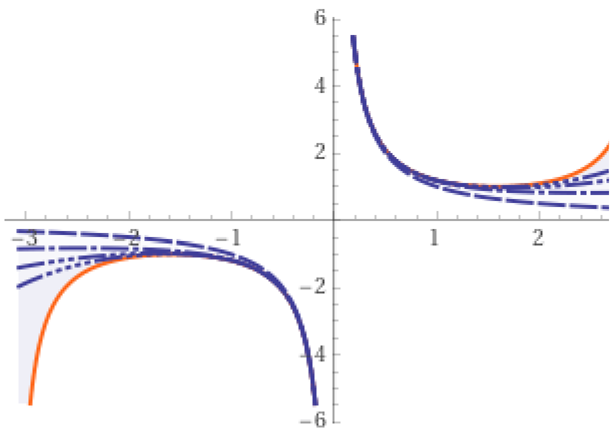
For function f = cot(x) and its taylor sereis is $\frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} + O(x^7)$ and the graphical visulaization of cot(x) taylor sereis with point zero and order 5. While other lines display for n orders, the red line represents the Taylor series' nature with point zero and order 5 as shown in figure 6. The Taylor series of cotx has a maximum positive value recorded of 4 and a range from a negative to a positive value at point 0 and order 5 with antisymmetry and both convergent and divergent.



(order n approximation shown with n dots)

Figure 5: Visualiztion of Taylor sereis of triognmitc fucntion cotx

For function f = cosec(x) and its taylor sereis is $\frac{1}{x}+\frac{x}{6}+\frac{7x^3}{360}+\frac{31x^5}{15120}+O(x^7)$ and the graphical visulaization of cosec(x) taylor sereis with point zero and order 5. While other lines display for n orders, the red line represents the Taylor series' nature with point zero and order 5 as shown in figure 6. The Taylor series of cosx has a maximum positive value recorded of 1 and a range from a negative to a positive value at point 0 and order 5 with symmetric nature and convergent and divergent.



(order n approximation shown with n dots)

Figure 6: Visualiztion of Taylor sereis of triognmitc fucntion cosecx

CONCLUSION

The Taylor series' trigonometric function behaves up to order five using Wolfram Mathematica software. Additionally, the tailored series of trigimatric functions, which varied in nature depending on the order, were obtained. The difference in the character of the series was also observed along with points for the same order. Every Taylor series of trigonometric functions has at least one point where the function is either convergent toward or divergent from the point, according to the process of observation. With the use of the Taylor Series, symmetric and antisymmetric nature was also investigated for tigronomic function.

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